Discrete Probability

• How can we obtain these probabilities p(s) ?

• The probability p(**s**) assigned to an outcome **s** equals the limit of the number of times **s** occurs divided by the number of times the experiment is performed.

• Once we know the probabilities p(s), we can compute the **probability of an event E** as follows:

• $p(E) = \sum_{s \in E} p(s)$

Discrete Probability

- **Example I:** A die is biased so that the number 3 appears twice as often as each other number.
- What are the probabilities of all possible outcomes?

- **Solution:** There are 6 possible outcomes $s_1, \ldots, s_6$.
- $p(s_1) = p(s_2) = p(s_4) = p(s_5) = p(s_6)$
- $p(s_3) = 2p(s_1)$
- Since the probabilities must add up to 1, we have:
- $5p(s_1) + 2p(s_1) = 1$
- $7p(s_1) = 1$
- $p(s_1) = p(s_2) = p(s_4) = p(s_5) = p(s_6) = 1/7$, $p(s_3) = 2/7$

Discrete Probability

- **Example II:** For the biased die from Example I, what is the probability that an odd number appears when we roll the die?

- **Solution:**
- $E_{odd} = \{s_1, s_3, s_5\}$
- Remember the formula $p(E) = \sum_{s \in E} p(s)$.
- $p(E_{odd}) = \sum_{s \in E_{odd}} p(s) = p(s_1) + p(s_3) + p(s_5)$
- $p(E_{odd}) = 1/7 + 2/7 + 1/7 = 4/7 = 57.14\%$

Conditional Probability

- If we toss a coin three times, what is the probability that an odd number of tails appears **(event E)**, if the first toss is a tail **(event F)** ?
- If the first toss is a tail, the possible sequences are TTT, TTH, THT, and THH.
- In two out of these four cases, there is an odd number of tails.
- Therefore, the probability of E, under the condition that F occurs, is 0.5.
- We call this **conditional probability**.

Conditional Probability

- If we want to compute the conditional probability of E given F, we use F as the sample space.
- For any outcome of E to occur under the condition that F also occurs, this outcome must also be in
$E \cap F$.

- **Definition:** Let E and F be events with $p(F) > 0$.
The conditional probability of E given F, denoted by $p(E \mid F)$, is defined as

- $p(E \mid F) = p(E \cap F)/p(F)$

Conditional Probability

- **Example:** What is the probability of a random bit string of length four contains at least two consecutive 0s, given that its first bit is a 0 ?

- **Solution:**
- E: “bit string contains at least two consecutive 0s”
- F: “first bit of the string is a 0”

- We know the formula $p(E \mid F) = p(E \cap F)/p(F)$.

- $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$
- $p(E \cap F) = 5/16$
- $p(F) = 8/16 = 1/2$
- $p(E \mid F) = (5/16)/(1/2) = 10/16 = 5/8 = 0.625$

Independence

- Let us return to the example of tossing a coin three times.
- Does the probability of event E (odd number of tails) **depend** on the occurrence of event F (first toss is a tail) ?
- In other words, is it the case that
$p(E \mid F) \neq p(E)$?
- We actually find that $p(E \mid F) = 0.5$ and $p(E) = 0.5$,
so we say that E and F are **independent events**.

Independence

- Because we have $p(E \mid F) = p(E \cap F)/p(F)$,
$p(E \mid F) = p(E)$ if and only if $p(E \cap F) = p(E)p(F)$.

- **Definition:** The events E and F are said to be independent if and only if $p(E \cap F) = p(E)p(F)$.

- Obviously, this definition is **symmetrical** for E and F. If we have $p(E \cap F) = p(E)p(F)$, then it is also true that $p(F \mid E) = p(F)$.

Independence

- **Example:** Suppose E is the event that a randomly generated bit string of length four begins with a 1, and F is the event that a randomly generated bit string contains an even number of 0s. Are E and F independent?

- **Solution:** Obviously, $p(E) = p(F) = 0.5$.

- $E \cap F = \{1111, 1001, 1010, 1100\}$
- $p(E \cap F) = 0.25$
- $p(E \cap F) = p(E)p(F)$

- Conclusion: E And F are **independent**.

Bernoulli Trials

- Suppose an experiment with **two possible outcomes**, such as tossing a coin.
- Each performance of such an experiment is called a **Bernoulli trial**.
- We will call the two possible outcomes a **success** or a **failure**, respectively.
- If p is the probability of a success and q is the probability of a failure, it is obvious that
$p + q = 1$.

Bernoulli Trials

- Often we are interested in the probability of **exactly k successes** when an experiment consists of **n independent Bernoulli trials**.

- **Example:**

A coin is biased so that the probability of head is 2/3. What is the probability of exactly four heads to come up when the coin is tossed seven times?

Bernoulli Trials

- **Solution:**
- There are $2^7 = 128$ possible outcomes.
- The number of possibilities for four heads among the seven trials is $C(7, 4)$.
- The seven trials are independent, so the probability of each of these outcomes is $(2/3)^4(1/3)^3$.
- Consequently, the probability of exactly four heads to appear is
- $C(7, 4)(2/3)^4(1/3)^3 = 560/2187 = 25.61\%$

Bernoulli Trials

- **Theorem:** The probability of k successes in n independent Bernoulli trials, with probability of success p and probability of failure q = 1 – p, is
- $C(n, k)p^k q^{n-k}$.
- See the textbook for the proof.
- We denote by b(k; n, p) the probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure q = 1 – p.
- Considered as function of k, we call b the **binomial distribution**.